Math 70 12.5 Natural Logs, Common Logs, and the Change-of-Base Formula

Lesson 40–41 (1.5 lessons)

Objectives

1) Use notation for common logs
   $\log(x)$ means $\log_{10}(x)$

2) Use notation for natural logs
   $\ln(x)$ means $\log_e(x)$

3) Understand the irrational number e

4) Calculate approximate values for common logs and natural logs.

5) Graph common logs and natural logs

6) Use the change-of-base formula
   - to calculate $\log_b(a)$ where $b \neq 10, e$
   - to graph $\log_b x = y$ where $b \neq 10, e$

B 12.5 ≈ M-G 9.7

abbreviation meaning

Notation: $\log x$ means $\log_{10} x$ and is called the common log. This log is the $\boxed{\text{LOG}}$ button on GC
If there's no base written, there is still a base! You assume base 10.

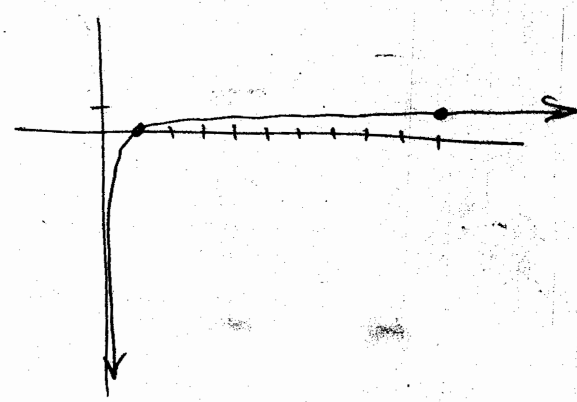
① Approximate $\log 7$ to nearest ten-thousandth.

$= 0.84509$ exact.

$\approx \boxed{0.8451}$ ← approximate answer

This means $10\hat{}1 = 10 < 7$ $10\hat{}2 = 100 > 7$ $10\hat{}^{.85} \approx 7$

② Graph $y_1 = \log x$ on GC



Notation: $\ln x$ means $\log_e x$ and is called the natural log. This is the $\boxed{\text{LN}}$ button on GC.

Above the $\boxed{\text{LN}}$, the second function is e^x.
e is an irrational number.

Math 70

Approximate to nearest ten-thousandth using GC

(3) e

$\underbrace{\boxed{\text{2nd}}\ \boxed{\text{LN}}}_{e^{\wedge}}\ \underbrace{\boxed{1}}_{e^1}\ \boxed{\text{ENTER}}$

$= 2.71828$

$\approx \boxed{2.7183}$ ← approximate

e ← exact

e to 25 decimal places

$e \approx 2.7182818284590452353603\ldots$

Δs

(4) e^2

$\boxed{\text{2nd}}\ \boxed{\text{LN}}\ \boxed{2}\ \boxed{\text{ENTER}}$

$= 7.38905$

$\approx \boxed{7.3891}$ ← approximate

e^2 ← exact

e is an irrational number!

- its decimal does not terminate
- its decimal does not repeat
- if we need an exact answer, we will write an expression using e.

No (5) e^3

$= 20.08553$

$\approx \boxed{20.0855}$ ← approximate

e^3 ← exact

(6) e^{-1}

$= .36787$

$\approx \boxed{.3679}$ ← approximate

e^{-1} ← exact

(7) $\ln 7$

$\boxed{\text{LN}}\ 7\ \boxed{)}\ \boxed{\text{ENTER}}$

$= 1.94591$

$\approx \boxed{1.9459}$ ← approximate

$\ln 7$ ← exact

This means

$e^1 = e \approx 2.7$

$e^2 \approx 7.4$

bigger base, smaller exponent

⇓

$10^{\sim .85} = 7$

then $e^{\sim 1.9} \approx 7$

⇑

smaller base, bigger exponent

(8) $\ln 100$

$= 4.60517$

$\approx \boxed{4.6052}$ ← approximate

$\ln 100$ ← exact

$\ln 7$ and $\ln 100$ are also irrational numbers w/ decimals that do not terminate or repeat.

- if we need an exact answer, we write a simplified log expression

Recall Inverse functions un-do each other.

$$f(f^{-1}(x)) = x$$
$$f^{-1}(f(x)) = x$$

Recall: $\left.\begin{aligned} f(x) &= b^x \\ f^{-1}(x) &= \log_b x \end{aligned}\right\}$ are inverse functions. for any valid base b.

If base is 10:

$\left.\begin{aligned} f(x) &= 10^x \\ f^{-1}(x) &= \log x \end{aligned}\right\}$ inverse functions

If base is e:

$\left.\begin{aligned} f(x) &= e^x \\ f^{-1}(x) &= \ln x \end{aligned}\right\}$ inverse functions

If we compose $f(f^{-1}(x)) = \boxed{10^{\log x} = x}$

or $\boxed{e^{\ln x} = x}$

If we compose $f^{-1}(f(x)) = \boxed{\log 10^x = x}$

or $\boxed{\ln e^x = x}$

inverse properties for base 10 and base e.

In particular, this means that

$\boxed{\log 10 = 1}$ because $10^1 = 10$

$\boxed{\ln e = 1}$ because $e^1 = e$

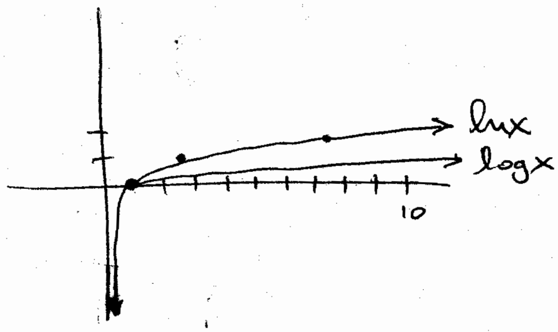
⑨ ln 0

= $\boxed{\text{undefined}}$

⑩ ln 1

= $\boxed{0}$

⑪ Graph y = ln x. How is it different from graph of log x?



The base e is between 2 and 3 so its y values are higher than base 10 log.

⑫ Approximate $\log_2 3 = x$ to 4 decimal places.

[Technically $x = \log_2 3$ is "Solved". That's the exact answer.]

step 1: Write equivalent exponential equation

$$2^x = 3$$

step 2: Take common logs of both sides of equation.

$$\log 2^x = \log 3$$

step 3: Use property of logs on LHS

$$x \cdot \log 2 = \log 3$$

$\boxed{\text{NOTE}}$: log 2 is a number — a constant! So is log 3! So we can isolate x by dividing both sides by that number

$$\frac{x \cdot \log 2}{\log 2} = \frac{\log 3}{\log 2}$$

CAUTION $\frac{\log 3}{\log 2} \neq \log \frac{3}{2}$

Since $\log \frac{3}{2} = \log 3 - \log 2$

$$\log_2 3 = x = \frac{\log 3}{\log 2} \quad \text{exact}$$

1.58496

$\approx \boxed{1.5850}$ approx

Math 70

In problem (12), we use common logs because they're on the GC. But we could have used natural logs. Does the result change?

(12) $\log_2 3 = x$ (It's already solved for x!)

$2^x = 3$

$\ln 2^x = \ln 3$

$x \ln 2 = \ln 3$

$\boxed{x = \log_2 3} = \boxed{x = \dfrac{\ln 3}{\ln 2}}$ exact

exact

1.58496

$= \boxed{1.5850}$ approx

Solve ÷ logs (2 logs)

vs.

The exact expression is the same, but base e.
The approximate result is identical.

log prop ÷ arguments (1 log)

CAUTION
log properties are different from change of base
$\log_b a - \log_b c = \log_b\left(\frac{a}{c}\right)$
$\log 3 - \log 2 = \log\left(\frac{3}{2}\right)$
$\log\left(\frac{3}{2}\right) \approx .1761$

If we'd used base 7 we'd get:

$$\log_2 3 = \frac{\log_7 3}{\log_7 2}$$

or any other base of logarithm.

Change of Base Formula

$\log_b x = \dfrac{\log x}{\log b}$ to change base b to base 10

$\log_b x = \dfrac{\ln x}{\ln b}$ to change base b to base e

$\log_b x = \dfrac{\log_c x}{\log_c b}$ to change base b to base c

Math 70

Q. What is the difference, if any, between these?

a) $\frac{\log 3}{\log 2}$ b) $\log\left(\frac{3}{2}\right)$ c) $\frac{\log 3}{2}$

A: They are all different from each other!

a) $\frac{\log 3}{\log 2} = \log_2 3$ change of base formula

$\approx \underline{\underline{1.58496}}$

b) $\log\left(\frac{3}{2}\right) = \log(1.5)$ or $\log(3) - \log(2)$ log property $\log_b\left(\frac{a}{c}\right) = \log_b a - \log_b c$

$\approx \underline{\underline{0.17609}}$

c) $\frac{\log 3}{2} = \frac{\log(3)}{2} = \frac{1}{2}\log(3) = \log 3^{1/2} = \log\sqrt{3}$

arithmetic coefficient log property $k\log_b a = \log_b a^k$ ½ exponent means square root.

$\approx \underline{\underline{0.23856}}$

Solve by brain, not GC:

(12) $\log 10 = x$

$10^x = 10$

$\boxed{x = 1}$

(13) $\log 10^5 = x$

inverse property

$\boxed{x = 5}$

✓ (14) $\log \frac{1}{10} = x$

$\boxed{x = -1}$

✓ (15) $\log \sqrt{10} = x$

$\boxed{x = \frac{1}{2}}$

✓ (16) $10^{\log 6} = x$

$\boxed{x = 6}$

inverse property

(17) $\log x = -3$

equivalent exponential

$10^{-3} = x$

$\boxed{x = .001}$

✓ (18) $\log x = \frac{1}{3}$

$10^{1/3} = x$

$\boxed{x = \sqrt[3]{10}}$

Math 70

Find approximate values. Round to nearest ten-thousandth.

✓ (13) $\log_5 7 = \dfrac{\log 7}{\log 5} \approx \boxed{1.2091}$

(14) $\log_2 1 = \boxed{0}$ (log property!)

✓ (15) $\log_2 10 = \dfrac{\log 10}{\log 2} = \boxed{3.3219}$ not the same as $\log_{10} 2$!

✓ (16) Use GC to look at graph of $y = \log_2 x$.

$Y_1 = \log(x)/\log(2)$

– OR –

$Y_1 = \ln(x)/\ln(2)$.

✓ (17) Calculate $\log \sqrt[3]{10}$

a) without GC

$\log \sqrt[3]{10} = \log_{10} 10^{1/3} = \boxed{\frac{1}{3}}$ inverse property $\log_b b^x = x$

b) with GC using cube root $\log(\sqrt[3]{10})$

$\boxed{\text{LOG}}$ $\boxed{\text{MATH}}$ 4. 10)) MATH 1.

$\uparrow$ $\sqrt[3]{\ }$ $\uparrow$ >frac

log($\sqrt[3]{\ }$(10)) > frac

c) with GC using $\frac{1}{3}$ power $\log(10^{1/3})$

log(10^(1/3)) > frac $= \boxed{\frac{1}{3}}$

CAUTION: NOT log(10)^(1/3) $\Rightarrow$ order of operations
nested parentheses inside out

Solve by brain, not GC.

(19) $\ln e^3 = x$

$e^x = e^3$

$\boxed{x = 3}$

✓ (20) $\ln \sqrt[5]{e} = x$

$e^x = \sqrt[5]{e}$

$\boxed{x = 1/5}$

✓ (21) $e^{\ln 2} = x$ inverse property

$\boxed{x = 2}$

✓ (22) $\log_{\sqrt{2}} x = 4$

$2^4 = x$

$\boxed{x = 16}$

✓ (23) $\log_x 2 = 3$

$x^3 = 2$

$\boxed{x = \sqrt[3]{2}}$

Solve each equation a) exactly
b) approximately, to 4 decimal places

✓ (24) $\log x = 1.2$

a) $\boxed{10^{1.2} = x}$ ← exact

$10^{1.2}$ is an irrational number: its decimal does not repeat or terminate. We write $10^{1.2}$ if we need an exact answer.

b) $x = 15.84893$

$\boxed{x \approx 15.8489}$ ← approximate

(25) $\ln 3x = 5$

a) $e^5 = 3x$

$\boxed{x = \frac{e^5}{3}}$ exact

b) $\frac{e^5}{3} \approx 49.47105$

$\boxed{49.4711}$ approx

(26) $\ln 5x = 8$

a) $e^8 = 5x$

$\boxed{x = \frac{e^8}{5}}$ exact

b) $\frac{e^8}{5} \approx 596.19159$

$= \boxed{596.1916}$ approx.